Electronic oscillator circuit

Solution: *a. The equation of motion of the damped and forced system is,*

$$I_L + I_R + I_C = L^{-1}\int_0^{t'} U dt + R^{-1}U + C\dot{U} = I_0 e^{i\omega t} = I_F \ ,$$

or

$$L^{-1}U + R^{-1}\dot{U} + C\ddot{U} = i\omega I_0 e^{i\omega t} \ .$$

b. Without source and without resistance the differential equation would be,

$$\ddot{U} + (LC)^{-1}U = 0 \ .$$

Therefore, the natural frequency would be,

$$\omega_0 = \frac{1}{\sqrt{LC}} \ .$$

c. Without source but with resistance the differential equation would be,

$$\ddot{U} + (RC)^{-1}\dot{U} + (LC)^{-1}U = 0 \ .$$

Therefore, the oscillation frequency would be,

$$\omega_\gamma = \sqrt{\omega_0^2 - \gamma^2} \ ,$$

com $2\gamma \equiv (RC)^{-1}$.
d. Inserting the ansatz $U = U_0 e^{i\omega t + i\phi}$,

$$L^{-1}U_0 e^{i\phi} + i\omega R^{-1}U_0 e^{i\phi} - \omega^2 C U_0 e^{i\phi} = i\omega I_0 \ ,$$

or

$$-\omega U_0 + 2i\gamma\omega U_0 + \omega_0^2 U_0 = ie^{-i\phi}\omega I_0/C \ .$$

e. The solutions are,

$$Z \equiv \frac{U_0}{I_0}e^{i\phi} = \frac{i\omega}{L^{-1} + i\omega R^{-1} - \omega^2 C} = \frac{1}{R^{-1} + i[\omega C - 1/(\omega L)]} = \frac{i\omega/C}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \ ,$$
$$|Z| = \left|\frac{U_0}{I_0}\right| = \frac{1}{\sqrt{R^{-2} + [\omega C - 1/(\omega L)]^2}} = \frac{\omega^2/C}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2\omega^2}}$$
$$\tan\phi = \frac{Im\ Z}{Re\ Z} = \frac{-[\omega C - 1/(\omega L)]}{R^{-1}} = R/(\omega L) - \omega RC = \frac{\omega_0^2 - \omega^2}{2\gamma\omega} \ .$$