Swing modes

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Solution: *a. Be x_1 and x_2 the independent rest positions of the two springs, such that the independent restorative forces are,*

$$F_1 = -k_1(x - x_1) \qquad and \qquad -F_2 = -k_2(x_2 - x) \ .$$

The resting position of the springs acting jointly x_0 follows from,

$$0 = F_1 + F_2 = -k_1(x_0 - x_1) - k_2(x_0 - x_2) \ ,$$

giving,

$$x_0 = \frac{k_1x_1 + k_2x_2}{k_1 + k_2} \ .$$

The equation of motion is now,

$$M\ddot{x} = F_1 + F_2 = -k_1(x_0 - x_1) - k_2(x_0 - x_2) = -(k_1 + k_2)(x - x_0) \ .$$

Hence,

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{M}} \ .$$

b. The same argument can be used to treat springs in parallel. We have $k = k_1 + k_2$, hence,

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{M}}.$$

c. For springs in series the derivation is more complicated. We call y and y_1, respectively, the current position and the resting position of the connection point between the springs and x and x_2, respectively, the current position and the resting position of the mass. Since the connection point has no mass, we have an equilibrium between the forces exerted by the two springs,

$$0 = F_1 + F_2 = -k_1(y - y_1) + k_2(x - x_2 - y + y_1) \ ,$$

yielding,

$$y - y_1 = \frac{k_2}{k_1 + k_2}(x - x_2) \ .$$

The equation of motion is now,

$$M\ddot{x} = F_M = -F_2 = -k_2(x - x_2 - y + y_1) = -\frac{k_1k_2}{k_1 + k_2}(x - x_2) \ .$$

Hence, $k = (k_1^{-1} + k_2^{-1})^{-1}$ e,

$$\omega_0 = \sqrt{\frac{(k_1^{-1} + k_2^{-1})^{-1}}{M}} \ .$$