Normal modes of two spring-coupled masses

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Solution: *a. For a chain of two masses we have,*

$$m_1\ddot{x}_1 = -k(x_1 - x_2) \quad , \quad m_2\ddot{x}_2 = -k(x_2 - x_1) \ .$$

Doing the ansatz $x_n = a_n e^{i\omega t}$ *and with the abbreviation* $\omega_j \equiv \frac{k}{m_j}$ *we get the characteristic equations,*

$$-\omega^2 a_1 = -\omega_1^2(a_1 - a_2) \quad , \quad -\omega^2 a_2 = -\omega_2^2(a_2 - a_1) \ .$$

b. The matrix form of the characteristic equations is,

$$\begin{pmatrix} \omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & \omega_2^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \ .$$

Solving,

$$0 = \det(\hat{M} - \omega^2 \mathbf{1}) = -\omega_1^2\omega^2 - \omega^2\omega_2^2 + \omega^4 = \omega^2(\omega^2 - \omega_1^2 - \omega_2^2) \ ,$$

we get the eigenvalues $\omega = 0$ *and* $\omega = \sqrt{\omega_1^2 + \omega_2^2}$.
c. The first eigenvector is obtained by,

$$\hat{M}\vec{a} = \begin{pmatrix} \omega_1^2 a_1 - \omega_1^2 a_2 \\ -\omega_2^2 a_1 + \omega_2^2 a_2 \end{pmatrix} = 0 \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \vec{a} \ ,$$

which is only possible when $a_1 = a_2$. *For the second eigenvalue,*

$$\hat{M}\vec{a} = \begin{pmatrix} \omega_1^2 a_1 - \omega_1^2 a_2 \\ -\omega_2^2 a_1 + \omega_2^2 a_2 \end{pmatrix} = (\omega_1^2 + \omega_2^2) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \omega^2 \vec{a} \ ,$$

which implies $\omega_1^2 a_2 = -\omega_2^2 a_1$. *Therefore, the normal modes are,*

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{a} = \frac{1}{\sqrt{\omega_1^4 + \omega_2^4}} \begin{pmatrix} \omega_1^2 \\ -\omega_2^2 \end{pmatrix} \ .$$

d. The center of mass is at the position $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$. *We call* $x_{rl} = x_1 - x_2$ *the relative coordinate. The equations are,*

$$\ddot{x}_{cm} = \frac{m_1\ddot{x}_1 + m_2\ddot{x}_2}{m_1 + m_2} = \frac{-k(x_1 - x_2) - k(x_2 - x_1)}{m_1 + m_2} = 0$$
$$\ddot{x}_{rl} = \ddot{x}_1 - \ddot{x}_2 = -\frac{k}{m_1}(x_1 - x_2) + \frac{k}{m_2}(x_2 - x_1) = -\frac{k}{\mu}(x_1 - x_2) = -(\omega_1^2 + \omega_2^2)(x_1 - x_2) \ ,$$

with $\mu^{-1} \equiv m_1^{-1} + m_2^{-1}$. *The first mode corresponds to a translation without vibration, the second to an anti-symmetrical vibration without translation around the center-of-mass. This corresponds to the results obtained by the reduced mass method.*