Spring-coupled chain of masses

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Solution: *a. For each mass we have,*

$$m\ddot{x}_n = -k(x_n - x_{n-1}) - k(x_n - x_{n+1}) \ .$$

Making the ansatz $x_n = A_n e^{i\omega t}$ *we get the characteristic equation,*

$$-\omega^2 A_n = -\omega_0^2(A_n - A_{n-1}) - \omega_0^2(A_n - A_{n+1}) \ .$$

In matrix notation,

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 & 0 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 & 0 & 0 \\ 0 & -\omega_0^2 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 2\omega_0^2 & -\omega_0^2 \\ 0 & 0 & 0 & -\omega_0^2 & \omega_0^2 \end{pmatrix} \ .$$

b. For a chain of three masses we have,

$$\begin{pmatrix} \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & \omega_0 \end{pmatrix} \ .$$

To find the eigenvalues, we solve,

$$0 = \det[\hat{M} - \lambda\mathbf{1}]] = -3\omega_0^4\lambda + 4\omega_0^2\lambda^2 - \lambda^3 = -\lambda(\lambda - \omega_0)(\lambda - 3\omega_0^2) \ ,$$

giving the eigenvalues, $\lambda = 0$, $\lambda = \omega_0^2$, *and* $\lambda = 3\omega_0^2$. *To find the eigenvectors, we do* $\hat{M}\vec{a} = 0\vec{a}$ *with* $\vec{a} \equiv (a_1, a_2, a_3)$, *yielding,*

$$a_1 = a_2 = a_3 \ ,$$

and therefore the eigenvector,

$$\vec{a} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ .$$

For the second eigenvector, we do $\hat{M}\vec{a} = \omega_0^2\vec{a}$ *yielding,*

$$a_2 = 0 \quad , \quad a_3 = -a_1 \ ,$$

and therefore the eigenvector,

$$\vec{a} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \ .$$

For the third eigenvector, we do $\hat{M}\vec{a} = 3\omega_0^2\vec{a}$ *yielding,*

$$a_1 - a_2 = 3a_1 \quad , \quad a_1 - a_2 = 3a_1 \quad , \quad -a_2 + a_3 = 3a_3 \ ,$$

or

$$a_1 = a_3 = -\tfrac{1}{2}a_2 \qquad , \qquad a_2 = a_2 \ ,$$

and therefore the eigenvector,

$$\vec{a} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\frac{1}{2}a_2 \\ a_2 \\ -\frac{1}{2}a_2 \end{pmatrix} / \frac{a_2}{\sqrt{2}} = \sqrt{\frac{2}{3}} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} \ .$$