Physical pendulum

Philippe W. Courteille, 05/02/2021

Solution: *a. We calculate the tensor of inertia I_{ij} for a coordinate system in the center of the cube:*

$$I_{ij} = \int_V \rho(\mathbf{r}) \left[\delta_{ij} \sum_{\alpha=1}^{3} x_\alpha^2 - x_i x_j \right] dV \ .$$

Hence,

$$\begin{aligned} I_{11} &= \int_V \rho(\vec{r}) \left[x_1^2 + x_2^2 + x_3^2 - x_1 x_1 \right] dV = \rho \int_{-d/2}^{d/2} (x_2^2 + x_3^2) dx_1 dx_2 dx_3 \\ &= \frac{\rho d^5}{6} = \frac{Md^2}{6} = I_{22} = I_{33} \\ I_{\imath \neq j} &= \int_V \rho(\mathbf{r}) x_i x_j dV = \rho \int_{-d/2}^{d/2} x_i x_j dx_i dx_j dx_k = 0 \ . \end{aligned}$$

Thus, I^{CM} adopts the form:

$$I^{CM} = \begin{pmatrix} Md^2/6 & 0 & 0 \\ 0 & Md^2/6 & 0 \\ 0 & 0 & Md^2/6 \end{pmatrix} \ .$$

Now we nail the cube to the wall. Here, the vertical distance b from the new axis of rotation $\vec{\omega} = \omega \hat{\mathbf{e}}_3$ until the center of mass is given by,

$$\mathbf{b} = \sqrt{2} \left(\frac{d}{2} - a \right) \hat{\mathbf{e}}_b \ .$$

$\hat{\mathbf{e}}_b$ shows from the origin of the new coordinate system (around which the rotation occurs) to the origin of the old coordinate system (the center-of-mass system), $\hat{\mathbf{e}}_b = -\hat{\mathbf{e}}_1 \sin\phi - \hat{\mathbf{e}}_2 \cos\phi$. With Steiner's theorem, $I_\omega = I_{\omega_0}^{CM} + Mb^2$, we get the inertial moment regarding the new rotation axis $I_\omega = 2M \left(\frac{1}{3} d^2 - da + a^2 \right)$. The torque is,

$$\mathbf{D} = \frac{\partial}{\partial t} \mathbf{L} = \frac{\partial}{\partial t} I \vec{\omega} = I_\omega \ddot{\phi} \hat{\mathbf{e}}_3 \ .$$

The force of gravitational attraction $F_g = -Mg\hat{e}_2$ cause, when displaced from the rest position, a torque,

$$\mathbf{D} = \mathbf{b} \times \mathbf{F}_g = -b\hat{\mathbf{e}}_b \times Mg\hat{\mathbf{e}}_2 = bMg \sin\phi \hat{\mathbf{e}}_3 \ .$$

Where ϕ is the angle between $\vec{b}$ and the axis $\hat{\mathbf{e}}_2$, that is, the direction of gravitational attraction. Since the torque has to be conserved ($\frac{\partial}{\partial t} \mathbf{L} = 0$), the sum of the torques must be zero,

$$0 = I_\omega \ddot{\phi} + bMg \sin\phi \ .$$