# Oscillation of a half cylinder

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**Solution:** *The mass of the half cylinder is,*

$$M = \int dm = \rho_0 \int_0^D \int_0^\pi \int_0^R \rho_0 r dr d\theta dz = \rho_0 \pi D \tfrac{R^2}{2} \ .$$

*The moment of inertia about the symmetry axis of the cylinder, if it were complete, would be,*

$$I_0 = \int r^2 dm = \rho_0 \int_0^D \int_0^\pi \int_0^R r^2 \rho_0 r dr d\theta dz = \rho_0 \pi D \tfrac{R^4}{4} = MR^2/2 \ .$$

*Following Steiner's theorem the moment of inertia with respect to support point of the half-cylinder is,*

$$I = I_0 + MR^2 = \tfrac{3}{2} MR^2 \ .$$

*We calculate the center-of-mass of a half cylinder lying on its flat side. The definition of the center-of-mass is,*

$$\mathbf{r}_{cm} = \frac{1}{M} \int \mathbf{r} dm \ .$$

*For symmetry reasons* $x_{cm} = 0$*. Also,*

$$y_{cm} = \frac{\rho_0}{M} \int_0^D \int_0^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} y dx dy dz = \frac{\rho_0 D}{M} \int_0^R 2y\sqrt{R^2-y^2} dy = \frac{\rho_0 D}{M} \int_0^{R^2} \sqrt{R^2-u} du$$
$$= -\frac{2}{3} \frac{\rho_0 D}{M} (R^2-u)^{\frac{3}{2}} \Big|_0^{R^2} = \frac{2}{3} R^3 \frac{\rho_0 D}{M} = \frac{4R}{3\pi} \ .$$

*Let* $\theta$ *be the angle of oscillation. Seen from the support point the center-of-mass is at,*

$$\mathbf{r} = \begin{pmatrix} 0 \\ R \\ 0 \end{pmatrix} - \begin{pmatrix} y_{cm} \sin\theta \\ y_{cm} \cos\theta \\ 0 \end{pmatrix} \ .$$

*The equation of motion is,*

$$m\ddot{x} = -F_{at} \ldots$$
$$I\ddot{\theta} = \vec{\tau} = \mathbf{r} \times \vec{F} = \begin{pmatrix} -y_{cm} \sin\theta \\ R - y_{cm} \cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -gM \\ 0 \end{pmatrix} = -y_{cm} g M \hat{e}_z \sin\theta$$
$$\ddot{\theta} = -\frac{y_{cm} g M}{I} \sin\theta = -\frac{\frac{4R}{3\pi} gM}{\frac{3}{2} MR^2} \sin\theta = -\frac{8g}{9\pi R} \sin\theta \ .$$

*Hence, the period of oscillation is,*

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{8g}{9\pi R}}} = \sqrt{\frac{9\pi^3}{2}} \sqrt{\frac{R}{g}} \ .$$