Normal modes on a string

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Solution: *a. The function describing the n-th mode can be written as,*

$$Y(x,t) = \sin\frac{n\pi x}{L}\cos\omega_n t \ .$$

The kinetic energy of each mass element of the wire is,

$$\frac{d}{dt}Y(x,t) = -A_1\omega_n \sin\frac{n\pi x}{L}\sin\omega_n t \ .$$

At the moments when the rope is stretched, $\omega_n t = \pi/2 + N\pi$, the energy is,

$$\frac{d}{dt}Y(x,t) = -A_1\omega_n \sin\frac{n\pi x}{L} \ .$$

The spatial integral gives the total energy,

$$E = \int_0^L \frac{m/L}{2}\left(\frac{d}{dt}Y(x,t)\right)^2 dx = A_1^2\omega_1^2\frac{m}{2L}\frac{L}{n\pi}\int_0^{n\pi}\sin^2\xi d\xi = A_1^2\omega_1^2\frac{m}{2L}\frac{n\pi}{2} = A_1^2\frac{m\omega_n^2}{4} \ .$$

With

$$\omega_n = 2\pi\frac{c}{\lambda_n} = 2\pi\frac{\sqrt{T/(m/L)}}{2L/n} = \pi\sqrt{\frac{Tn^2}{mL}} \ ,$$

we have,

$$E = \frac{A_1^2 n^2\pi^2 T}{4L} \ .$$

b. We consider the stationary wave at time, $t = 0$. At this instant, ...