Dispersion near an atomic resonance

Philippe W. Courteille, 05/02/2021

Solution: *Expressing k by ω,*

$$k = \frac{\omega n(\omega)}{c} = \frac{\omega}{c}\left(1 - \frac{\alpha}{\omega^2 - \omega_0^2}\right)$$

we can calculate the phase velocity by,

$$v_f = \frac{c}{n} = \frac{c}{1 - \frac{\alpha}{\omega^2-\omega_0^2}} \ .$$

we can calculate the group velocity by,

$$v_g = \frac{d\omega}{dk}\bigg|_{k=k_l} = \frac{1}{\frac{dk}{d\omega}\big|_{\omega=\omega_l}} = \frac{1}{\frac{d}{d\omega}\frac{\omega n(k)}{c}\Big|_{\omega=\omega_l}} = \frac{1}{\frac{d}{d\omega}\frac{\omega}{c}\left(1 - \frac{\alpha}{\omega^2-\omega_0^2}\right)\Big|_{\omega=\omega_l}}$$
$$= \frac{c}{1 + \alpha\frac{\omega_l^2+\omega_0^2}{(\omega_l^2-\omega_0^2)^2}} = \frac{c}{n - (n-1)\frac{2\omega^2}{\omega^2-\omega_0^2}} \ .$$

For $\omega \simeq \omega_0$ we can approximate,

$$v_g = \frac{c}{1 + \frac{\alpha}{2(\omega_l-\omega_0)^2}} \ .$$

Alternatively, approximating $\omega \simeq \omega_0$ we can calculate,

$$k \simeq \frac{\omega}{c}\left(1 - \frac{\alpha}{2\omega_0(\omega-\omega_0)}\right) \ ,$$

and express ω by k,

$$\omega = \tfrac{1}{2}\left(\omega_0 + \tfrac{\alpha}{2\omega_0} + ck\right) \pm \sqrt{\tfrac{1}{4}\left(\omega_0 + \tfrac{\alpha}{2\omega_0} + ck\right)^2 - ck\omega_0} \ ,$$

giving,

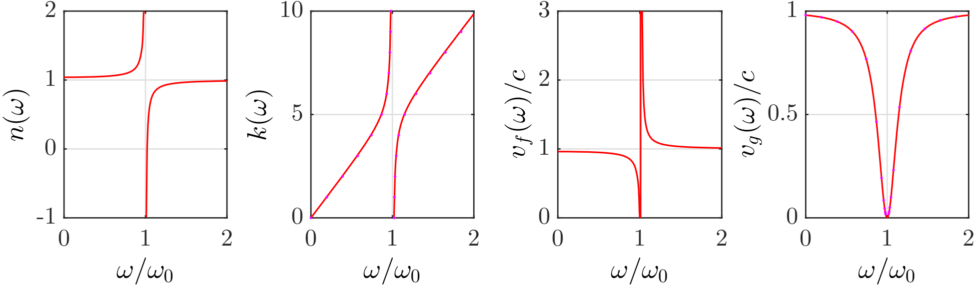
$$v_g = \frac{d\omega}{dk}\bigg|_{k=k_l} = \tfrac{c}{2} \pm \tfrac{c}{2}\frac{-\omega_0 + \frac{\alpha}{2\omega_0} + ck_l}{\sqrt{\left(\omega_0 + \frac{\alpha}{2\omega_0} + ck_l\right)^2 - 4ck_l\omega_0}} \ .$$

The graph shows the approximation as points in magenta. Similarly, we can show that for $n(\omega) = 1 + \frac{\alpha}{1+(\omega-\omega_0)^2/\Gamma^2}$ group velocity is,

$$v_g = \frac{c}{n - (n-1)\frac{2\omega^2(\omega-\omega_0)}{\Gamma^2+(\omega-\omega_0)^2}} \ ,$$

and for $n(\omega) = 1 + \alpha e^{-(\omega-\omega_0)^2/\Gamma^2}$ the group velocity is,

$$v_g = \frac{c}{n - (n-1)\frac{2\omega(\omega-\omega_0)}{\Gamma^2}} \ .$$



$n(\omega)$
$k(\omega)$
$v_f(\omega)/c$
$v_g(\omega)/c$
ω/ω_0
ω/ω_0
ω/ω_0
ω/ω_0