

Gauge of retarded potentials

Solution: From the retarded potentials,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\varrho(\mathbf{r}', t_r)}{R} d^3r' \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_r)}{R} d^3r'$$

we calculate the derivatives,

$$\frac{d}{dt}\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\dot{\varrho}(\mathbf{r}', t_r)}{R} d^3r' \quad \text{and} \quad \nabla \cdot \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \nabla \cdot \frac{\mathbf{j}(\mathbf{r}', t_r)}{R} d^3r' .$$

To evaluate the divergence of the vector potential we first calculate,

$$\nabla \cdot \frac{\mathbf{j}}{R} - \frac{1}{R} \nabla \cdot \mathbf{j} = \mathbf{j} \cdot \nabla \frac{1}{R} = -\mathbf{j} \cdot \nabla' \frac{1}{R} = -\nabla' \cdot \frac{\mathbf{j}(\mathbf{r}', t - R/c)}{R} + \frac{1}{R} \nabla' \cdot \mathbf{j} .$$

We now calculate the divergence of the current,

$$\nabla' \cdot \mathbf{j}(\mathbf{r}', t - R/c) = \frac{\partial \mathbf{j}}{\partial \mathbf{r}'} + \frac{\partial \mathbf{j}}{\partial t_r} \cdot \nabla' t_r = \frac{\partial \mathbf{j}}{\partial \mathbf{r}'} - \frac{\partial \mathbf{j}}{\partial t_r} \cdot \nabla t_r = \frac{\partial \mathbf{j}}{\partial \mathbf{r}'} - \nabla \cdot \mathbf{j}(\mathbf{r}', t - R/c) ,$$

where we defined the explicit spatial derivative, $\frac{\partial \mathbf{j}}{\partial \mathbf{r}'}$. We conclude,

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \nabla \cdot \frac{\mathbf{j}(\mathbf{r}', t_r)}{R} d^3r' \\ &= \frac{\mu_0}{4\pi} \int \left[-\nabla' \cdot \frac{\mathbf{j}(\mathbf{r}', t - R/c)}{R} + \frac{1}{R} \nabla' \cdot \mathbf{j} + \frac{1}{R} \nabla \cdot \mathbf{j} \right] d^3r' \\ &= \frac{\mu_0}{4\pi} \oint \cancel{\frac{\mathbf{j}(\mathbf{r}', t - R/c)}{R}} \cdot d\mathbf{S}^0 + \frac{\mu_0}{4\pi} \int \frac{1}{R} \cancel{\frac{\partial \mathbf{j}}{\partial \mathbf{r}'}}^{\neg \dot{}} d^3r' = -\epsilon_0 \mu_0 \frac{d}{dt} \Phi(\mathbf{r}, t) . \end{aligned}$$