Stationary solution of the Bloch equations

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Solution: *Starting from the Bloch equations,*

$$0=\frac{d}{dt}\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ \tilde\rho_{12}\\ \tilde\rho_{21}\end{pmatrix}=\begin{pmatrix}0 & \Gamma & \frac{\imath}{2}\Omega & -\frac{\imath}{2}\Omega\\ 0 & -\Gamma & -\frac{\imath}{2}\Omega & \frac{\imath}{2}\Omega\\ \frac{\imath}{2}\Omega & -\frac{\imath}{2}\Omega & \imath\Delta-\gamma & 0\\ -\frac{\imath}{2}\Omega & \frac{\imath}{2}\Omega & 0 & -\imath\Delta-\gamma\end{pmatrix}\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ \tilde\rho_{12}\\ \tilde\rho_{21}\end{pmatrix}~,$$

we find from the first and third line using the normalization condition,

$$0=\Gamma\rho_{22}+\tfrac{\imath}{2}\Omega\tilde\rho_{12}-\tfrac{\imath}{2}\Omega\tilde\rho_{21}\qquad\textit{and}\qquad 0=\tfrac{\imath}{2}\Omega-\imath\Omega\rho_{22}+(\imath\Delta-\gamma)\tilde\rho_{12}~.$$

Solving,

$$\rho_{22}=\frac{\imath\Omega}{2\Gamma}(\tilde\rho_{21}-\tilde\rho_{12})\qquad\textit{and}\qquad\tilde\rho_{12}=\frac{\imath\Omega}{2(\imath\Delta-\gamma)}(2\rho_{22}-1)~.$$

Hence,

$$\rho_{22}=\frac{\Omega^2}{4\Gamma}\left(\frac{1}{\imath\Delta+\gamma}+\frac{1}{-\imath\Delta+\gamma}\right)(1-2\rho_{22})=\frac{\frac{\gamma}{2\Gamma}\Omega^2}{\Delta^2+\frac{\gamma}{\Gamma}\Omega^2+\gamma^2}$$

$$\tilde\rho_{12}=\frac{\imath\Omega}{2(\imath\Delta-\gamma)}\left(\frac{\frac{\gamma}{\Gamma}\Omega^2}{\Delta^2+\frac{\gamma}{\Gamma}\Omega^2+\gamma^2}-1\right)=\frac{\frac{\Omega}{2}(-\Delta+\imath\gamma)}{\Delta^2+\frac{\gamma}{\Gamma}\Omega^2+\gamma^2}~.$$

For $\gamma=\frac{\Gamma}{2}$ *we get,*

$$\rho_{22}=\frac{\frac{1}{4}\Omega^2}{\Delta^2+\frac{1}{2}\Omega^2+\frac{1}{4}\Gamma^2}\qquad\textit{and}\qquad\tilde\rho_{12}=\frac{\frac{\Omega}{2}(-\Delta+\frac{\imath}{2}\Gamma)}{\Delta^2+\frac{1}{2}\Omega^2+\frac{1}{4}\Gamma^2}~.$$

For $\gamma=\frac{\Gamma}{2}+\beta\gg\frac{\Gamma}{2}$ *we get,*

$$\rho_{22}\simeq\frac{\frac{\beta}{2\Gamma}\Omega^2}{\Delta^2+\beta^2}\stackrel{\Delta\to0}{\longrightarrow}\frac{\Omega^2}{2\Gamma\beta}\qquad\textit{and}\qquad\tilde\rho_{12}\simeq\frac{\frac{\Omega}{2}(-\Delta+\imath\beta)}{\Delta^2+\beta^2}\stackrel{\Delta\to0}{\longrightarrow}\frac{\imath\Omega}{2\beta}~.$$