Thermal population of a harmonic oscillator

Philippe W. Courteille, 27/12/2021

Solution: *The density operator,*

$$\hat{\rho} = \sum_n P_n |n\rangle\langle n| \ ,$$

satisfies $\hat{\rho}^2 \neq \hat{\rho}$ *e* $Tr\ \hat{\rho} = 1$. *It allows to calculate the most likely value for the population and is obtained using the rule* $\sum_{n=0}^{\infty} U^n = (1-U)^{-1}$ *with the abbreviation* $U \equiv e^{-\hbar\omega/k_BT}$,

$$\begin{aligned}
\langle\hat{n}\rangle &= Tr\ \hat{\rho}\hat{n} = \sum_m \langle m|\hat{\rho}\hat{n}|m\rangle = \sum_m \langle m|\sum_n P_n|n\rangle\langle n|\hat{n}|m\rangle = \sum_m mP_m\langle m|m\rangle \\
&= \sum_m \frac{me^{-m\beta\hbar\omega}}{\sum_n e^{-n\beta\hbar\omega}} = (1-e^{-\beta\hbar\omega})\sum_m me^{-m\beta\hbar\omega} = (1-e^{-\beta\hbar\omega})\frac{-1}{\hbar\omega}\frac{\partial}{\partial\beta}\sum_m e^{-m\beta\hbar\omega} \\
&= (1-e^{-\beta\hbar\omega})\frac{-1}{\hbar\omega}\frac{\partial}{\partial\beta}\frac{1}{1-e^{-\beta\hbar\omega}} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1} \ .
\end{aligned}$$

For the average energy, with $E_n = n\hbar\omega$,

$$\langle\hat{E}\rangle = \sum_n E_n p_n = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \ .$$

This is precisely the distribution proposed by Planck for the light modes in the black-body radiator. We can now express the occupation probability of state n *as,*

$$P_n = \frac{e^{-n\beta\hbar\omega}}{\sum_m e^{-m\beta\hbar\omega}} = (1-e^{-\beta\hbar\omega})e^{-n\beta\hbar\omega} = \frac{\langle\hat{n}\rangle^n}{(1+\langle\hat{n}\rangle)^{n+1}} \ .$$