Bloch equations for three levels

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Solution: *Let us first verify that the Bloch equations for two levels follow the formula (2.99). The Hamiltonian is* $\hat{H} = \hbar\omega_0|2\rangle\langle 2| + \frac{\hbar}{2}\Omega e^{-\imath\omega t}|1\rangle\langle 2| + \frac{\hbar}{2}\Omega e^{\imath\omega t}|2\rangle\langle 1|$ *the operator of Liouville is*

$$\begin{aligned}\mathcal{L}\rho &= \sum\nolimits_{i,j}\gamma_{ij}\Big(|j\rangle\langle i|\sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|i\rangle\langle j| - \sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|i\rangle\langle j|j\rangle\langle i| \\ &+ |j\rangle\langle i|\sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|i\rangle\langle j| - |i\rangle\langle j|j\rangle\langle i|\sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|\Big) \\ &= \gamma\Big(|2\rangle\langle 1|\sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|1\rangle\langle 2| - \sum\nolimits_{m,n}|n\rangle\rho_{nm}\langle m|1\rangle\langle 2|2\rangle\langle 1| \\ &+ |2\rangle\langle 1|\sum\nolimits_{m,n}|m\rangle\rho_{mn}\langle n|1\rangle\langle 2| - |1\rangle\langle 2|2\rangle\langle 1|\sum\nolimits_{m,n}|n\rangle\rho_{nm}\langle m|\Big) \\ &= \gamma\Big(2|2\rangle\rho_{11}\langle 2| - \sum\nolimits_{n}|n\rangle\rho_{n1}\langle 1| - \sum\nolimits_{m}|1\rangle\rho_{m1}\langle m|\Big)\ .\end{aligned}$$

Follows the master equation,

$$\begin{aligned}\frac{d}{dt}\rho &= -\frac{\imath}{\hbar}[\hat{H},\rho] + \mathcal{L}\rho \\ &= -\imath\begin{pmatrix}\frac{1}{2}\Omega e^{-\imath\omega t}\rho_{21} - \frac{1}{2}\Omega e^{\imath\omega t}\rho_{12} & \frac{1}{2}\Omega e^{-\imath\omega t}\rho_{22} - \frac{1}{2}\Omega e^{-\imath\omega t}\rho_{11} - \omega_0\rho_{12} \\ \frac{1}{2}\Omega e^{\imath\omega t}\rho_{11} - \frac{1}{2}\Omega e^{\imath\omega t}\rho_{22} + \omega_0\rho_{21} & \frac{1}{2}\Omega e^{\imath\omega t}\rho_{12} - \frac{1}{2}\Omega e^{-\imath\omega t}\rho_{21}\end{pmatrix} + \gamma\begin{pmatrix}2\rho_{22} & -\rho_{12} \\ -\rho_{21} & -2\rho_{22}\end{pmatrix} \\ &= -\imath\begin{pmatrix}\frac{\Omega}{2}\tilde{\rho}_{21} - \frac{\Omega}{2}\tilde{\rho}_{12} & \frac{\Omega}{2}e^{-\imath\omega t}\rho_{22} - \frac{\Omega}{2}\rho_{11}e^{-\imath\omega t} - \omega_0\tilde{\rho}_{12}e^{-\imath\omega t} \\ \frac{\Omega}{2}e^{\imath\omega t}\rho_{11} - \frac{\Omega}{2}e^{\imath\omega t}\rho_{22} + \omega_0\tilde{\rho}_{21}e^{\imath\omega t} & \frac{\Omega}{2}\tilde{\rho}_{12} - \frac{1}{2}\Omega\tilde{\rho}_{21}\end{pmatrix} + \gamma\begin{pmatrix}2\rho_{22} & -\rho_{12} \\ -\rho_{21} & -2\rho_{22}\end{pmatrix}\end{aligned}$$

transforming to the rotating frame, $\tilde{\rho}_{12}e^{-\imath\omega t} = \rho_{12}$,

$$\frac{d}{dt}\tilde{\rho} = \begin{pmatrix}-\frac{\imath}{2}\Omega\tilde{\rho}_{21} + \frac{\imath}{2}\Omega\tilde{\rho}_{12} & -\frac{\imath}{2}\Omega\rho_{22} + \frac{\imath}{2}\Omega\rho_{11} - \imath\Delta\tilde{\rho}_{12} \\ -\frac{\imath}{2}\Omega\rho_{11} + \frac{\imath}{2}\Omega\rho_{22} + \imath\Delta\tilde{\rho}_{21} & -\frac{\imath}{2}\Omega\tilde{\rho}_{12} + \frac{\imath}{2}\Omega\tilde{\rho}_{21}\end{pmatrix} + \gamma\begin{pmatrix}2\rho_{22} & -\rho_{12} \\ -\rho_{21} & -2\rho_{22}\end{pmatrix}\ .$$

Moving to the Bloch vector notation we find the well-known expression (2.75).
We now generalize to three levels using the equations (2.100). For three levels the Hamiltonian is,

$$\hat{H} = \hbar\omega_2|2\rangle\langle 2| + \hbar\omega_3|3\rangle\langle 3| + \frac{\hbar}{2}\Omega_{12}e^{-\imath\omega_a t}|1\rangle\langle 2| + \frac{\hbar}{2}\Omega_{12}e^{\imath\omega_a t}|2\rangle\langle 1| + \frac{\hbar}{2}\Omega_{23}e^{\imath\omega_b t}|2\rangle\langle 3| + \frac{\hbar}{2}\Omega_{23}e^{-\imath\omega_b t}|3\rangle\langle 2|\ ,$$

and the Liouville operator is,

$$\begin{aligned}\mathcal{L}\rho &= \gamma_{12}\left(2|1\rangle\rho_{22}\langle 1| - 2|2\rangle\rho_{22}\langle 2| - |1\rangle\rho_{12}\langle 2| - |2\rangle\rho_{21}\langle 1|\right) \\ &+ \gamma_{23}\left(2|3\rangle\rho_{22}\langle 3| - 2|2\rangle\rho_{22}\langle 2| - |2\rangle\rho_{23}\langle 3| - |3\rangle\rho_{32}\langle 2|\right) \\ &+ \gamma_{13}\left(2|1\rangle\rho_{33}\langle 1| - 2|3\rangle\rho_{33}\langle 3| - |1\rangle\rho_{13}\langle 3| - |3\rangle\rho_{31}\langle 1|\right)\ .\end{aligned}$$

Follows the master equation,

$$\begin{aligned}\frac{d}{dt}\rho &= -\frac{\imath}{\hbar}[\hat{H},\rho] + \mathcal{L}\rho \\ &= -\imath\left[\begin{pmatrix} 0 & \frac{\Omega_{12}}{2}e^{-\imath\omega_a t} & 0 \\ \frac{\Omega_{12}}{2}e^{\imath\omega_a t} & -\imath\omega_2 & -\frac{\imath\Omega_{23}}{2}e^{\imath\omega_b t} \\ 0 & \frac{\Omega_{23}}{2}e^{-\imath\omega_b t} & -\imath\omega_3 \end{pmatrix}, \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}\right] \\ &+ \begin{pmatrix} 2\gamma_{12}\rho_{22} + 2\gamma_{13}\rho_{33} & -\gamma_{12}\rho_{12} & -\gamma_{13}\rho_{13} \\ -\gamma_{12}\rho_{21} & -2(\gamma_{12}+\gamma_{23})\rho_{22} & -\gamma_{23}\rho_{23} \\ -\gamma_{13}\rho_{31} & -\gamma_{23}\rho_{32} & 2\gamma_{23}\rho_{22} - 2\gamma_{13}\rho_{33} \end{pmatrix}\end{aligned}$$

Using MAPLE we derive the Bloch equations,

$$\begin{aligned}\dot{\rho}_{11} &= \tfrac{\imath}{2}\Omega_{12}\left(e^{\imath\omega_a t}\rho_{12} - e^{-\imath\omega_a t}\rho_{21}\right) + 2\gamma_{12}\rho_{22} + 2\gamma_{13}\rho_{33} = 1 - \rho_{22} - \rho_{33} \\ \dot{\rho}_{22} &= -\tfrac{\imath}{2}\Omega_{12}\left(e^{\imath\omega_a t}\rho_{12} - e^{-\imath\omega_a t}\rho_{21}\right) + \tfrac{\imath}{2}\Omega_{23}\left(e^{-\imath\omega_b t}\rho_{23} - e^{\imath\omega_b t}\rho_{32}\right) - 2\gamma_{12}\rho_{22} - 2\gamma_{23}\rho_{22} \\ \dot{\rho}_{12} &= \tfrac{\imath}{2}\Omega_{12}e^{-\imath\omega_a t}(\rho_{11} - \rho_{22}) + \tfrac{\imath}{2}\Omega_{23}e^{-\imath\omega_b t}\rho_{13} + (\imath\omega_2 - \gamma_{12})\rho_{12} = \dot{\rho}_{21}^* \\ \dot{\rho}_{13} &= \tfrac{\imath}{2}\Omega_{23}e^{-\imath\omega_b t}\rho_{12} - \tfrac{\imath}{2}\Omega_{12}e^{-\imath\omega_a t}\rho_{23} + (\imath\omega_3 - \gamma_{13})\rho_{13} = \dot{\rho}_{31}^* \\ \dot{\rho}_{23} &= -\tfrac{\imath}{2}\Omega_{12}e^{\imath\omega_a t}\rho_{13} + \tfrac{\imath}{2}\Omega_{23}e^{\imath\omega_b t}(\rho_{22} - \rho_{33}) + (\imath\omega_3 - \imath\omega_2 - \gamma_{23})\rho_{23} = \dot{\rho}_{32}^* \ .\end{aligned}$$

This result can be cast into a matrix form.