Optical density of a cold cloud

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Solution: *The optical density with Doppler broadening is,*

$$OD(T,\omega) = Ln(T)(\sigma \star \rho)(\omega) = Ln(T)\int_{-\infty}^{\infty}\sigma(v)\rho(v)dv$$
$$= L\frac{P}{k_BT}\sqrt{\frac{m}{2\pi k_BT}}\frac{6\pi}{k^2}\int_{-\infty}^{\infty}e^{-mv^2/2k_BT}\frac{\Gamma^2}{4(\omega-\omega_0-kv)^2+\Gamma^2}dv \ .$$

The Doppler limit yields,

$$k\bar{v} = k\sqrt{\frac{k_BT_D}{m}} = k\sqrt{\frac{\hbar\Gamma}{m}} \ ,$$

where $\bar{v} = \sqrt{k_BT/m}$ *is the mean atomic velocity (or the* rms*-width) of the Maxwell distribution.*

a. For the transition at 461 nm we get $k\bar{v} = 5.2$ *MHz, which is much narrower than the natural linewidth. Therefore, we can approximate the Gaussian by the* δ*-distribution:*

$$\int_{-\infty}^{\infty}e^{-mv^2/2k_BT}dv = \sqrt{\frac{2k_BT}{m}}\int_{-\infty}^{\infty}e^{-x^2}dx = \sqrt{\frac{2\pi k_BT}{m}} = \int_{-\infty}^{\infty}\sqrt{\frac{2\pi k_BT}{m}}\delta(kv)d(kv) \ .$$

Hence,

$$OD_{461}(T_D,\omega) = Ln(T_D)\sqrt{\frac{m}{2\pi k_BT_D}}\frac{6\pi}{k^2}\int_{-\infty}^{\infty}\frac{\Gamma_{461}^2}{4(\omega-\omega_0-kv)^2+\Gamma_{461}^2}\sqrt{\frac{2\pi k_BT_D}{m}}\delta(kv)d(kv)$$
$$= Ln(T_D)\frac{6\pi}{k^2}\frac{\Gamma_{461}^2}{4(\omega-\omega_0)^2+\Gamma_{461}^2} \ ,$$

which means that we retrieve the default expression for optical density,

$$OD_{461}(0,\omega) = Ln(T_D)\sigma(T_D) \ .$$

b. The natural width of the transition at 689 nm is much narrower than the Doppler width. Therefore, we can approximate the Lorentzian by the δ*-distribution:*

$$\int_{-\infty}^{\infty}\frac{\Gamma_{689}^2}{4(\omega-\omega_0-kv)^2+\Gamma_{689}^2}dv = \frac{\Gamma_{689}}{k}\int_{-\infty}^{\infty}\frac{dx}{1+4x^2} = \frac{\pi\Gamma_{689}}{2k} = \int_{-\infty}^{\infty}\frac{\pi\Gamma_{689}}{2k}\delta(\omega-\omega_0-kv)d(kv) \ .$$

Hence,

$$OD_{689}(T_D,\omega) = Ln(T_D)\sqrt{\frac{m}{2\pi k_BT_D}}\frac{6\pi}{k^2}\int_{-\infty}^{\infty}e^{-mv^2/2k_BT_D}\frac{\pi\Gamma_{689}}{2k}\delta(\omega-\omega_0-kv)d(kv)$$
$$= Ln(T_D)\sqrt{\frac{m}{2\pi k_BT_D}}\frac{6\pi}{k^2}\frac{\pi\Gamma_{689}}{2k}e^{-m(\omega-\omega_0)^2/2k_BT_Dk^2} \ .$$

In resonance,

$$OD_{689}(T_D,\omega_0) = Ln(T_D)\sqrt{\frac{\pi}{2}}\frac{\sigma_0\Gamma_{689}}{2k\bar{v}_D} \ ,$$

The fraction $\Gamma_{689}/2k\bar{v}_D$ *can be interpreted as the spectral overlap between the Lorentzian-shaped absorption profile and the Maxwell distribution.*
c. The ratio is,

$$\frac{OD_{689}(T_D,\omega_0)}{OD_{461}(T_D,\omega_0)} = \sqrt{\frac{\pi}{8}}\frac{\Gamma_{689}}{k\bar{v}_D} \ .$$