Purity of two-level atoms with spontaneous emission

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Solution: *From* $1 = \rho_{11}(0) + \rho_{22}(0)$ *follows that* $Tr\ \rho = 1$ *always. With*

$$\hat{\rho}^2 = \begin{pmatrix} \rho_{11}^2 + \rho_{12}\rho_{21} & \rho_{11}\rho_{12} + \rho_{12}\rho_{22} \\ \rho_{11}\rho_{21} + \rho_{21}\rho_{22} & \rho_{12}\rho_{21} + \rho_{22}^2 \end{pmatrix}$$

follows

$$Tr\ \rho^2 = Tr\ \rho - 2\det\rho = 1 - 2\rho_{22} + 2\rho_{22}^2 + 2\rho_{12}\rho_{21}\ .$$

Inserting the stationary solution (2.75) of the Bloch equations,

$$\rho_{22} = \frac{\Omega^2}{4\Delta^2 + 2\Omega^2 + \Gamma^2} \qquad \textit{and} \qquad \rho_{12} = e^{\imath\Delta t}\frac{\Omega(2\Delta + \imath\Gamma)}{4\Delta^2 + 2\Omega^2 + \Gamma^2}$$

we calculate,

$$\begin{aligned} Tr\ \rho^2 &= 1 - \frac{2\Omega^2(4\Delta^2 + 2\Omega^2 + \Gamma^2)}{(4\Delta^2 + 2\Omega^2 + \Gamma^2)^2} + \frac{2\Omega^4}{(4\Delta^2 + 2\Omega^2 + \Gamma^2)^2} + \frac{2\Omega^2(4\Delta^2 + \Gamma^2)}{(4\Delta^2 + 2\Omega^2 + \Gamma^2)^2} \\ &= 1 - 2\left(\frac{\Omega^2}{4\Delta^2 + 2\Omega^2 + \Gamma^2}\right)^2 = 1 - 2\rho_{22}^2 < 1\ . \end{aligned}$$

These quantities measure the purity of the state.