Optical forces applied to atomic cooling

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Abstract

We present a deduction for the radiative pressure force and the dipole gradient force in two-level atoms using the density operator formalism. Furthermore, we show how these forces are applied in the atomic cooling exemplifying with two transitions of the $^{164}\text{Dy}$ atoms. The optical forces are the key to the cooling experiments in which they are fundamental to the operation of the standard apparatus like Zeeman slower, magneto-optical trap and optical-dipole trap.

I. Interactions between two-level atoms and light

The interaction between atoms and light[1], for many cases, causes transitions between only two quantum states$^1$, in this context, a two-level system is a good approximation for an atom whose Hamiltonian is given by

$$\hat{H}_{\text{atom}} = \hbar \omega_1 \hat{\sigma}^- \hat{\sigma}^+ + \hbar \omega_2 \hat{\sigma}^+ \hat{\sigma}^-,$$

where $\hat{\sigma}^\pm$ are the Pauli ladder operators that acts on the space $\{|1\rangle \text{ (ground state)}, |2\rangle \text{ (excited state)}\}$. Moreover, the atom is well characterized by the density operator $\hat{\rho}$ given by

$$\hat{\rho} = \frac{1 - p}{2} \hat{\sigma}^- \hat{\sigma}^+ + \frac{1 + p}{2} \hat{\sigma}^+ \hat{\sigma}^- + q \hat{\sigma}^- \hat{\sigma}^+ + q^* \hat{\sigma}^+ \hat{\sigma}^-,$$

where $p$ is the population inversion and $q$ is the coherence. The electric field $\mathbf{E}$ from light is usually much large than magnetic field $\mathbf{B}$ ($|\mathbf{E}| = c|\mathbf{B}|$), thus we can treat only electric interactions. Specifically, it is interesting the monochromatic waves given by

$$\mathbf{E} = \left( \frac{E_0(r)}{2} e^{-i\omega t} e^{i\mathbf{k}\mathbf{r}} + \frac{E_0^*(r)}{2} e^{i\omega t} e^{-i\mathbf{k}\mathbf{r}} \right) \mathbf{e},$$

where $\mathbf{e}$ is the polarization, $\omega$ is the angular frequency, $\mathbf{k}$ is the wavevector and $E_0(r)$ is the complex amplitude. In first approximation, the strongest interaction is the dipole interaction with Hamiltonian given by $\hat{H}_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}$, where $\mathbf{d} = \hat{\mu} \hat{\sigma}^- + \hat{\mu}^* \hat{\sigma}^+$ is the dipole moment operator$^2$ and $\hat{\mu} \equiv \langle 1 | \mathbf{d} | 2 \rangle$ is the transition dipole moment. In the Dirac picture$^3$, we have

$$\hat{H}_{\text{int}} = \hat{H}_{\text{slow}} + \hat{H}_{\text{fast}},$$

$$\hat{H}_{\text{slow}} = -\frac{\hbar}{2} \left[ \Omega e^{-i\Delta t} e^{i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^-} + \Omega^* e^{i\Delta t} e^{-i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^+} \right],$$

$$\hat{H}_{\text{fast}} = -\frac{\hbar}{2} \left[ \Omega e^{i(\omega + \omega_0)t} e^{-i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^-} + \Omega^* e^{-i(\omega + \omega_0)t} e^{i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^+} \right],$$

where $\hbar \Omega(r) \equiv (\hat{\mu} \cdot \mathbf{e})E_0(r)$ is the Rabi frequency, $\hbar \Omega(r) = (\hat{\mu} \cdot \mathbf{e})E_0^*(r)$ is the counter-rotating frequency, $\omega_0 \equiv \omega_2 - \omega_1$ is the angular resonance frequency and $\Delta \equiv \omega - \omega_0$ is the detuning. The component $\hat{H}_{\text{fast}}$ is negligible due to the fast oscillations ($\omega + \omega_0 \gg \Delta$), this is called Rotating Wave Approximation. Thus, we obtain

$$\hat{H}_{\text{int}} = -\frac{\hbar}{2} \left( \Omega e^{-i\Delta t} e^{i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^+} + \Omega^* e^{i\Delta t} e^{-i\mathbf{k}\mathbf{r} \cdot \hat{\sigma}^-} \right).$$

The time-evolution of $\hat{\rho}$ in the Dirac picture is given by the Liouville equation:

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}_0 \hat{\rho}, \quad \mathcal{L}_0 \hat{\rho} \equiv \frac{i}{\hbar} [\hat{\rho}, \hat{H}_{\text{int}}],$$

where $\mathcal{L}_0$ is the Liouville superoperator. This equation only describes stimulated absorption

$^1$There are phenomena like multi-photon absorption[2] in which it necessary to consider three or more quantum states

$^2$Atoms do not have permanent dipole moment due to their spherical symmetry, therefore $\langle n | \mathbf{d} | n \rangle = 0$.

$^3\hat{H}_{\text{int}} = \hat{U} \hat{H}_{\text{int}} \hat{U}$ with $\hat{U} = e^{-i\hat{H}_{\text{atom}}t/\hbar}$. 
and emission and excludes the spontaneous emission introduced by Albert Einstein with the phenomenological rates $A$ and $B$. It is necessary the quantization of the electromagnetic field to obtain the description of the Einstein rates, at which the electromagnetic vacuum can also interact with the atom. The complete equation is called master equation given by

$$\frac{d\hat{\rho}}{dt} = (\mathcal{L}_0 + \mathcal{L}_{sp})\hat{\rho},$$

$$\mathcal{L}_{sp}\hat{\rho} \equiv \frac{\Gamma}{2} \left( 2\hat{\sigma}^- \hat{\rho} \hat{\sigma}^+ - \hat{\sigma}^+ \hat{\rho} \hat{\sigma}^- - \hat{\rho} \hat{\sigma}^+ \hat{\sigma}^- \right),$$

where $\Gamma = A$ is the natural linewidth or spontaneous rate and $\mathcal{L}_{sp}$ is the Lindbladt super-operator. The unitary transformation $\hat{S} = \hat{\sigma}^{-} \hat{\sigma}^{+} + e^{-i\Delta t} \hat{\sigma}^{+} \hat{\sigma}^{-}$ is convenient because the transformed Hamiltonian $H = \hat{S}^\dagger \hat{H}_{int} \hat{S} + i\hbar (\partial_t \hat{S}^\dagger) \hat{S}$ is time-independent. Thus, the matricial form of the master equation considering the dipole approximation ($e^{i\mathbf{k}\cdot\mathbf{r}} \simeq 1$) is given by\(^4\)

$$\frac{d\bar{\sigma}}{dt} = A\bar{\sigma} + \bar{a},$$

$$A = \begin{bmatrix} -\Gamma/2 & -\Delta & 0 \\ \Delta & -\Gamma/2 & \Omega \\ 0 & -\Omega & -\Gamma \end{bmatrix}, \quad \bar{a} = \begin{bmatrix} 0 \\ 0 \\ -\Gamma \end{bmatrix},$$

where $\bar{\sigma}$ is the Bloch vector given by

$$\bar{\sigma} = \begin{bmatrix} 2\Re(qe^{-i\Delta t}) \\ 2\Im(qe^{-i\Delta t}) \\ p' \end{bmatrix}, \quad q' = \langle \hat{\rho}|2 \rangle, \quad p' = p.$$

Solving $A\bar{\sigma}(\infty) = \bar{a}$ we obtain the stationary solution that is given by

$$p'_s = -\frac{1}{1 + s}, \quad q'_s = e^{i\Delta t} \left( \frac{\Delta}{\Omega} - i\frac{\Gamma}{2\Omega} \right) \frac{s}{1 + s},$$

where $p'_s = p'(\infty), \quad q'_s = q'(\infty)$ and $s$ is the saturation parameter given by

$$s \equiv \frac{2\Omega^2}{4\Delta^2 + \Gamma^2} = \frac{I/I_s}{1 + (2\Delta/\Gamma)^2}, \quad I_s = \frac{2\Omega^2}{\Gamma^2},$$

where $I_s$ is the saturation intensity and $I$ the light intensity.

\(^4\)We are considering $\Omega = \Omega^*$, because the phase factor given by $\Omega = |\Omega|e^{i\phi}$ can be incorporate in $q$.

II. Radiation pressure and dipole gradient forces

The Ehrenfest theorem defines the expectation force given by

$$\mathbf{F} = -\langle \nabla \hat{H}_{int} \rangle = -\text{Tr} \hat{\rho} \nabla \hat{H}_{int}$$

With the atom at origin $\mathbf{r} = 0$ we obtain the force $\mathbf{F} = \mathbf{F}_{sc} + \mathbf{F}_{dp}$ using the stationary solution $\hat{\rho}(\infty)$, where

$$\mathbf{F}_{rp} = \hbar \mathbf{k} \mathbf{R}_{scatt}, \quad R_{scatt} \equiv \frac{\Gamma}{2} \frac{s}{1 + s},$$

$$\mathbf{F}_{dp} = -\nabla U_{dp}, \quad U_{dp} \equiv \frac{\hbar \Delta}{2} \ln(1 + s).$$

The component $\mathbf{F}_{rp}$ is called radiation pressure force and it can be understood as a non-conservative force given by the photons scattering process as shown in the Figure (1). Furthermore, for $s \gg 1$, this force “saturates”, i.e. it is limited by the spontaneous rate $\Gamma$.

Figure 1: Illustration of the radiation pressure force. A photon is absorbed and the atom gains its momentum $\hbar \mathbf{k}$. Then, there is spontaneous emission at random direction isotropically. Consecutive absorption and emission happen such that, on average, the emissions do not change the expectation momentum is $\langle \mathbf{p}_{\text{atom}} \rangle = \hbar \mathbf{k}$.

The component $\mathbf{F}_{dp}$ is called dipole gradient force and it can be derived by the potential $U_{dp}$, i.e. we have a conservative force that is zero for resonant beams ($\Delta = 0$) or plane waves ($\nabla E_0 = 0$). It is common to use beams far from resonance such that $|\Delta| \gg \Gamma \Rightarrow s \simeq \Omega^2/(2\Delta^2) \Rightarrow s \ll 1$. With this, applying the
approximation $\ln(1 + s) \simeq s$, we obtain

$$U_{dp} \simeq \left( \frac{\hbar \Delta}{2} \right) s.$$ 

III. The cooling of atomic gases

The optical forces made possible the hard task of cooling and trapping atomic gases until critical temperatures at which we obtain a Bose-Einstein condensate (BEC). This achievement awarded several physicist with the Nobel prizes\[3\]. The standard process involves three apparatus that use the optical forces: (i) the Zeeman slower (ZS), (ii) the magneto-optical trap (MOT) and (iii) the optical-dipole trap (ODT). First of all, we create a hot atomic beam that will be slowed down by the ZS. The cold atomic beam will be trapped and cooled by the MOT becoming a ultra-cold cloud that will load the ODT. Then, through the forced evaporative cooling technique, we will reach the BEC. In the next sections, we will analysis how the optical forces is used in each apparatus exemplifying with two transitions $J = 8$ to $J = 9$ of $^{164}$Dy[4] shown in the table (1).

<table>
<thead>
<tr>
<th>-</th>
<th>Transition 1</th>
<th>Transition 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0$</td>
<td>626 nm</td>
<td>421 nm</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>854 kHz</td>
<td>202 MHz</td>
</tr>
<tr>
<td>$I_s$</td>
<td>72 $\mu W/cm^2$</td>
<td>56 mW/cm$^2$</td>
</tr>
</tbody>
</table>

Table 1: Relevant transitions of a dysprosium atom that we will use in the following sections, where $\lambda_0 = 2\pi c/\omega_0$ is the wavelength of resonance, $\Gamma$ is the natural linewidth and $I_s$ is the saturation intensity.

IV. Zeeman slower

The ZS shown in the figure (2) is an apparatus in which a hot atomic beam and a counterpropagating laser beam travels through a tapered solenoid that generated a varying magnetic field.

$$B(z) = B_0 \sqrt{1 - \frac{z}{l_0}} + B_{bias} ,$$

$$B_0 = \frac{\hbar v_0}{\lambda \mu'}, \ B_{bias} = \frac{h\delta}{\mu'} ,$$

where $B_{bias}$ calibrates the deceleration to a higher value removing the laser detuning $\delta$. In the figure (3) we have the magnetic field for $^{164}$Dy atom due transition 2, table (1).

5The quantum numbers $J$ and $J'$ are the total angular momentum.

6We are consider a slow spatial variation of the magnetic field such that the atom will "see" a constant field.
Figure 3: Magnetic field for a ZS with $^{164}$Dy atoms due the transition 2 using $m_J = 8 \ (g_J = 1.24)$ to $m'_J = 9 \ (g'_J = 1.22)$ and the capture velocity $v_0 = 645 \text{ m/s}$. In the graph (a) the detuning is $\delta = 0$ and in the graph (b) the detuning is $\delta = -18 \Gamma$.

V. Optical molasses technique

The optical molasses technique\cite{5} is a cooling method in which an atomic gas is cooled through the radiative pressure force. The apparatus consists of counter-propagating beams with the same intensities that will slow down the atoms in the intersect region\textsuperscript{7}. The detuning, due to the Doppler effect, is given by

$$\Delta = \delta - k \cdot v \ , \ \delta \equiv \omega - \omega_0 .$$

For simplicity, let us consider a unidimensional case\textsuperscript{8} shown in figure (4) with $v = v e_z$ and $k = k e_z$. Thus, if $v$ and $I/I_s$ are small, far away from saturation, the force $F_{rp}(v) = F_{rp}(v)e_z$ can be written by the Taylor expansion given by

$$F_{rp}(v) \simeq F_{rp}(0) + \left[ \frac{\partial F_{rp}}{\partial v}(0) \right] v .$$

The resulting force $F_{molasses} = F_{rp}(v) - F_{rp}(-v)$ given by the counter-propagating beams can be written as

$$F_{molasses} \simeq -\alpha v \ , \ \alpha = 4\hbar k^2 I \frac{\Gamma}{I_s [1 + I/I_s + (2\delta/\Gamma)^2]^2} .$$

For the red-detuned case, the constant $\alpha$ is positive and, therefore, the force $F_{molasses}$ works as a damping. The figure (5) shows the damping acceleration on $^{164}$Dy due the transition 1, table (1).

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The energy decay is given by

$$\frac{dE}{dt} = \frac{d}{dt} \left( \frac{mv^2}{2} \right) = \left( \frac{m}{dt} \right) v = -\alpha v^2 \Rightarrow \ E(t) = E(0)e^{-t/\tau_{damp}} \ , \ \tau_{damp} \equiv \frac{m}{2\alpha} .$$

where $\tau_{damp}$ is the damping time that usually is in order of microseconds. It may seem that we can slow down the atoms until to absolute zero, however the radiative pressure force $F_{rp}$ is an average such that it is submitted to fluctuations due to the photonic recoil rate of the stimulated absorption and spontaneous emission which will fixed a minimum temperature.

\textsuperscript{7}This method does not need of a atomic beam like the ZS.

\textsuperscript{8}We can easily extend this result for a generic velocity $v = (v_x, v_y, v_z)$.
given by

\[ T_D = \frac{\hbar \Gamma}{2k_B}, \]

where \( k_B \) is the Boltzmann constant. This temperature is called Doppler cooling limit.

VI. Magneto-optical trap

The optical molasses technique, discussed in the section (V), only slow down atoms in the intersect region of the laser beams, thus the cooling depends of the diffusion of these atoms which takes a considerable time. We can improve this method trapping the atoms close to this region, that is the idea of a MOT (Magneto-Optical Trap)[6]. The principle is introduce a linear spatial dependence in the radiation pressure force with a quadrupole magnetic field generated by coils in anti-Helmholtz\(^9\) configuration and light beams with polarization \( \sigma^\pm \) as shown in figure (6).

\[ \Delta \sigma^\pm = \delta \pm \delta^\pm. \]

Figure 6: Setup of a Magneto-optical trap.

Let us consider the simplest unidimensional case with the velocity \( v = ve_z \) a magnetic field given by

\[ B = B(z)e_z, \quad B(z) = B_0 ze_z, \]

where \( B_0 \) is constant. Now, we have to consider the total angular momentum \( J \) of our two-level atom, furthermore we will consider the simplest case \( J = 0 \rightarrow J' = 1 \) shown in figure (7). The magnetic field will break the degeneracy of \( J' = 1 \) into \( m_j' = 0, \pm 1 \) and it may seen that we can consider a two-level system anymore. However, due the selection rules, the polarized beam only excites transitions with \( \Delta m_j = \pm 1 \) such that we can consider a two-level-system for each probable transition, \( m_J = 0 \rightarrow m_j' = +1 \) and \( m_J = 0 \rightarrow m_j' = -1 \).

\[ \Delta \delta_{zee} = (g_j m_j' - g_j m_J) \mu_B \frac{\partial B}{\partial z}. \]

Zeeman shift

At \( z' = z > 0 \), the detuning of the \( \sigma^\pm \) transitions are given by \( \Delta \sigma^\pm = \delta \pm \delta^\pm \). Therefore, the probability to absorb the \( \sigma^- \) beam is greater (\( \Delta \sigma^- < \Delta \sigma^+ \)) and increases until the point with \( \delta = \delta^- \) whereas the probability to absorb the \( \sigma^+ \) decreases. Thus, the atom recoils in negative \( z \)-direction more times than it recoils in positive \( z \)-direction. The same analysis can be done at a point in \( z < 0 \). For a general \( z \), the atom "sees" each beam with a detuning given by

\[ \Delta \sigma^\pm = \delta \mp \delta_{zee} z - kv, \]

Figure 7: Scheme of a unidimensional MOT

The total force \( F_{\text{MOT}} = F_{\text{rp}}(\Delta \sigma^+) - F_{\text{rp}}(\Delta \sigma^-) \) can be written following the some process done in section (V) considering small \( v \) and \( z \), and the Taylor expansion for both variables. With this we have

\[ F_{\text{MOT}} = -\alpha v - \beta z, \quad \beta \equiv \frac{\alpha \delta_{zee}}{k} \Rightarrow \]

\[ \frac{d^2 z}{dt^2} + \left( \frac{1}{2\tau_{\text{damp}}} \right) \frac{dz}{dt} + (\omega_{\text{MOT}}^2) z = 0, \]
where $\omega_{MOT} = \sqrt{\beta/m}$. It is possible to extend this result for the three-dimensional case where we have a damped harmonic motion for each direction.

VII. Optical dipole trap

The ODTs\cite{7} are important tool to trap ultra-cold gases using the conservative dipole force. There are two types of traps: red-detuned ($\Delta < 0$) and blue-detuned ($\Delta > 0$). The red-detuned ODT is the simplest ODT that can be done with a focused Gaussian beam, however we can not remove completely the radiative pressure force, even for $\Delta \gg 0$.

Already the blue-detuned ODT is experimentally much more complicated and there are some methods to build one, the basic idea is to create potential walls such that the atoms will fill the "dark space", the advantage is that we can remove completely the radiative pressure force because there is no light in the dark space. In this section, we will analysis the red-detuning ODT using a Gaussian beam with power $P$ whose intensity in cylindrical coordinates $\{r, \phi, z\}$ is given by

$$I(r, z) = \frac{2P}{\pi w(z)^2} \exp \left(-\frac{2r^2}{w(z)^2}\right),$$

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2},$$

where $w(z)$ is the waist, $w_0$ is the minimal waist (minimum beam radius), $z_R = \pi w_0^2/\lambda$ is the Rayleigh length and $\lambda$ is the wavelength. The potential $U_{dp}$ can be written as

$$U_{dp}(r, z) \simeq \frac{\hbar \Gamma^2}{8\Delta I_s} I(r, z),$$

$$U_0 \equiv U_{dp}(0, 0) \simeq \frac{\hbar \Gamma^2}{8\Delta I_s} \frac{2P}{\pi w_0^2},$$

where $U_0$ is the minimum of the potential energy called trap depth at which it is possible to trap atoms with energies smaller than $|U_0|$. The figure (8) shows the potential energy $U_{dp}(r, 0)$ on a $^{164}Dy$ atom generated by the transition 1, table (1), and a focused beam with $P = 100$ mW and $w_0 = 41 \, \mu m$. In this case, the maximum temperature that it is possible to trap atoms is in order of $mK$.

Figure 8: Potential energy on $^{164}Dy$ atoms due the transition 1, table (1). We also has a focused beam with $P = 100$ mW and $w_0 = 41 \, \mu m$. The maximum $U_{dp}$ is reach for $\Delta = 1854\Gamma$.

For low temperatures, the potential $U_{dp}$ can be written using the Taylor expansion in two variables $(r, z)$ in which we obtain a harmonic approximation given by

$$U_{dp}(r, z) \simeq -U_0 + \frac{1}{2} m \left(\omega_r^2 r^2 + \omega_z^2 z^2\right),$$

$$\omega_r \equiv \sqrt{\frac{4U_0}{m w_0^2}}, \quad \omega_z \equiv \sqrt{\frac{2U_0}{m z_R^2}},$$

where $m$ is the atomic mass. In general, $z_R \gg w_0 \Rightarrow \omega_z \ll \omega_r$, that means the oscillation amplitude in $z$-direction is much larger than oscillation amplitude in radial direction, therefore to increase the trap efficiency, it is common to apply a second ODT. You could think that it is not possible cooling atoms with ODTs because we have only conservative forces, however if we slowly decrease the beams power, the faster atoms will scape from the trap and the gas will thermalize in a small temperature, this method is called forced evaporative cooling.
References


